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# CONDITIONS FOR THE EXISTENCE OF DISCONTINUITY SURFACES OF IRREVERSIBLE STRAINS IN ELASTOPLASTIC MEDIA

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*Constraints are obtained on the stresses of a plastically compressed elastoplastic medium at which the occurrence of discontinuities of irreversible strains is possible. The loading surfaces are taken to be piecewise linear closed surfaces. Velocities of motion of irreversible-strain discontinuity surfaces are calculated.*

**Key words:** *dynamics of elastoplastic media, irreversible compressibility, strain discontinuity, dissipative shock waves.*

**Introduction.** Existence conditions and propagation mechanism of discontinuity surfaces of irreversible strains (dissipative shock waves) are necessary elements in the formulation of boundary-value problems of irreversible deformation dynamics and are a subject of continuous interest [1–8]. In model relations it is difficult to use rough loading surfaces with irreversible changes in volumetric strains taken into account. Strain discontinuities that arise in this case are combined, and the conditions for their occurrence differ greatly depending on what points of the loading surface correspond to prestress. The present paper considers piecewise linear loading surfaces.

**1. Basic Relations of the Model.** Let the strains admitted by an elastoplastic medium be small, i.e., let the strain tensor components $e_{ij}$ be given by the relations

$$e_{ij} = (u_{i,j} + u_{j,i})/2 = e^e_{ij} + e^p_{ij}, \tag{1.1}$$

where $u_i$ are the components of the displacement vector of points of the medium and $e^e_{ij}$ and $e^p_{ij}$ are the reversible (elastic) component and irreversible (plastic) component of the total strain, respectively.

The stresses $\sigma_{ij}$ in an isotropic medium are related to the reversible strains $e^e_{ij}$ according to Hooke's law

$$\sigma_{ij} = \lambda e^e_{kk}\delta_{ij} + 2\mu e^e_{ij}.$$

Irreversible strains $e^p_{ij}$ begin to accumulate in a deformable medium when the stress states reach the loading surface:

$$f^{(s)}(\sigma_{ij}, \tau) = k, \qquad k = \text{const}, \quad s = 1, 2, \ \ldots \ . \tag{1.2}$$

The constant $k$ is usually identified with the yield point, and the strain history parameters $\tau_j$ are described by a kinetic equation, for example, in the form [9, 10]

$$\dot{\tau}_j = A^j_{em}\dot{e}^p_{em}. \tag{1.3}$$

Using the Mises maximum principle [relations (1.2) become a plastic potential in this case] and according to the associated plastic flow law, the plastic strain rate is given by the relation

$$\varepsilon^p_{ij} = \dot{e}^p_{ij} = \sum_{s=1}^{n} \psi_s \frac{\partial f^{(s)}}{\partial \sigma_{ij}}.$$

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Here $\psi_s > 0$ for $\dfrac{\partial f^{(s)}}{\partial \sigma_{ij}}\,\dot\sigma_{ij} > 0$ and $\psi_s = 0$ for $\dfrac{\partial f^{(s)}}{\partial \sigma_{ij}}\,\dot\sigma_{ij} \leqslant 0$.

Let the loading surface be given by the relations

$$\max|\sigma_i - \sigma_j|/2 + q\sigma = k \quad (i \neq j), \qquad \sigma + \varphi(\tau) = 0; \tag{1.4}$$

$$\max|\sigma_i - \sigma| + q\sigma = 2k/3, \qquad \sigma + \varphi(\tau) = 0, \tag{1.5}$$

where $\sigma_i$ are the principal values of the stress tensor, $\sigma = (\sigma_1 + \sigma_2 + \sigma_3)/3$, and $k$ and $q$ are constants of the elastoplastic medium; the function $\varphi(\tau)$ is specified on the basis of experimental data.

In the space of principal stresses, relations (1.4) specify the Coulomb–Mohr pyramid for which the shift of the base depends on the value of the single parameter of the strain history $\tau$. We note that in [10], this parameter was taken to be the density of the medium, and in [9], it was volumetric strain.

The loading surface (1.5) differs from a Coulomb–Mohr pyramid only in that its base is the hexagon of the maxim reduced shear stress rather than the hexagon of the maximum shear stress. Conditions of the maximum reduced shear stress were apparently first formulated by Ishlinskii [11]. The properties of model systems of equations obtained under these conditions were studied by Ivlev and coworkers [12]. Therefore, the plasticity condition (1.5) used in this paper will be called the Ishlinskii–Ivlev pyramid. As in ideal plasticity theory, which ignores irreversible volume changes, these piecewise linear loading surfaces have extreme properties [12, 13]. If, in the equations of these surfaces, the constants $k$ and $q$ are identical and the same experimental dependence $\varphi(\tau)$ is used, all the possible nonconcave loading surfaces are located between the Coulomb–Mohr and Ishlinskii–Ivlev pyramids. In particular, among such loading surfaces is the classical surface represented by the Mises–Schleicher cone.

In each flat region of the loading surfaces (1.4) and (1.5), as well as on the edges and at the angular points of these surfaces, the above relations, together with the equations of motion of an elastoplastic medium

$$\sigma_{ij,j} + \rho\chi_i = \rho\dot v_i \tag{1.6}$$

make closed systems of equations. In relations (1.6), $\chi_i$ are the body-force distribution densities and $v_i$ are the displacement velocity components of points of the medium ($v_i = \dot u_i$).

**2. Relations on Discontinuity Surfaces.** Let a strain-discontinuity surface $\Sigma(t)$ move at a constant velocity $c$ in a deformable elastoplastic medium which occupies volume $V$. The surface $\Sigma(t)$ divides the volume $V$ into two parts: $V_1$ and $V_2$ ($V = V_1 + V_2$); the surface $S$ bounding the volume $V$ is also divided into the parts $S_1$ and $S_2$ ($S = S_1 + S_2$). We assume that the surface $\Sigma(t)$ moves in the direction of its positive unit normal $\boldsymbol{\nu}$ from $V_1$ in $V_2$.

The parameters of the stress–strain state and motion of the points of the medium, which are continuous in $V_1$ and $V_2$ and undergo a discontinuity on $\Sigma(t)$, should satisfy the discontinuity compatibility conditions.

The conservation laws imply the following dynamic compatibility conditions for the stress discontinuities and displacement velocities:

$$[\sigma_{ij}]\nu_j = -\rho c[v_i], \qquad [\sigma_{ij}v_i]\nu_j + \rho c([v_iv_i] + 2[W])/2 = 0. \tag{2.1}$$

Here square brackets denote a jump of a quantity on the surface $\Sigma(t)$: $[m] = m^+ - m^-$ [$m^+$ and $m^-$ are the values of the quantity ahead of and behind the surface $\Sigma(t)$, respectively]; $W$ is the mass density of the internal-energy distribution. The kinematics of the moving discontinuity surfaces also imposes constraints on the possible discontinuities $[v_i] = -c[u_{i,j}]\nu_j$ (Hadamard condition).

According to the elastoplastic model, the medium must have only one source of internal energy — irreversible strain (the thermal conductivity of the medium is neglected). Then, for the discontinuity $[W]$, we obtain

$$\rho[W] = \frac{1}{2}\,[\sigma_{ij}e^e_{ij}] - \int\limits_{e^{p+}_{ij}}^{e^{p-}_{ij}} \sigma_{ij}\,de^p_{ij}. \tag{2.2}$$

In view of relation (2.2), the energy conservation law on $\Sigma(t)$ [the second condition in (2.1)] becomes

$$\frac{1}{2}\,\rho c[v_iv_i] + [\sigma_{ij}v_i]\nu_j + \frac{1}{2}\,c[\sigma_{ij}e^e_{ij}] - c\int\limits_{e^{p+}_{ij}}^{e^{p-}_{ij}} \sigma_{ij}\,de^p_{ij} = 0. \tag{2.3}$$

Using the kinematic compatibility condition for Hadamard discontinuities, from relation (1.1) we have

$$[e_{ij}] = -([v_i]\nu_j + [v_j]\nu_i)/(2c) = [e^e_{ij}] + [e^p_{ij}]. \tag{2.4}$$

According to the Betti theorem, the stress and reversible strains across the surface $\Sigma(t)$ are linked by the relation $\sigma^+_{ij}[e^e_{ij}] = e^{e+}_{ij}[\sigma_{ij}]$; with the use of the above relation, formula (2.4), and the first equality in (2.1), condition (2.3) is reduced to the form

$$\frac{\sigma^+_{ij} + \sigma^-_{ij}}{2}\,[e^p_{ij}] = -\int\limits_{e^{p+}_{ij}}^{e^{p-}_{ij}} \sigma_{ij}\, de^p_{ij}. \tag{2.5}$$

For the power of the surface forces, we write the following relations:

$$\int\limits_S \sigma_{ij} v_i N_j\, dS = \int\limits_{S_1} \sigma_{ij} v_i N_j\, dS + \int\limits_{S_2} \sigma_{ij} v_i N_j\, dS + \int\limits_\Sigma \sigma^-_{ij} v^-_i \nu_j\, dS - \int\limits_\Sigma \sigma^+_{ij} v^+_i \nu_j\, dS + \int\limits_\Sigma [\sigma_{ij} v_i]\nu_j\, dS$$

$$= \int\limits_{V_1} (\sigma_{ij} v_i)_{,j}\, dV + \int\limits_{V_2} (\sigma_{ij} v_i)_{,j}\, dV + \int\limits_\Sigma [\sigma_{ij} v_i]\nu_j\, dS = \int\limits_{V_1} \rho v_i \frac{\partial v_i}{\partial t}\, dV + \int\limits_{V_2} \rho v_i \frac{\partial v_i}{\partial t}\, dV + \int\limits_{V_1} \sigma_{ij} v_{i,j}\, dV + \int\limits_{V_2} \sigma_{ij} v_{i,j}\, dV$$

$$- \int\limits_{V_1} \rho\chi_i v_i\, dV - \int\limits_{V_2} \rho\chi_i v_i\, dV + \int\limits_\Sigma [\sigma_{ij} v_i]\nu_j\, dS = \frac{1}{2}\Big[\int\limits_{V_1} \frac{\partial}{\partial t}(\rho v_i v_i + \sigma_{ij} e^e_{ij})\, dV + \int\limits_{V_2} \frac{\partial}{\partial t}(\rho v_i v_i + \sigma_{ij} e^e_{ij})\, dV\Big]$$

$$+ \int\limits_{V_1} \sigma_{ij}\varepsilon^p_{ij}\, dV + \int\limits_{V_2} \sigma_{ij}\varepsilon^p_{ij}\, dV - \int\limits_V \rho\chi_i v_i\, dV + \int\limits_\Sigma [\sigma_{ij} v_i]\nu_j\, dS = \frac{d}{dt}\int\limits_V \frac{1}{2}(\rho v_i v_i + \sigma_{ij} e^e_{ij})\, dV$$

$$+ \int\limits_V \sigma_{ij}\varepsilon^p_{ij}\, dV - \int\limits_V \rho\chi_i v_i\, dV + \int\limits_\Sigma \Big(\rho c\,\frac{[v_i v_i]}{2} + c\,\frac{[\sigma_{ij} e^e_{ij}]}{2} + [\sigma_{ij} v_i]\nu_j\Big)\, dV.$$

Taking into account relations (2.3) and (2.5), we finally obtain

$$\frac{d}{dt}\int\limits_V \frac{1}{2}(\rho v_i v_i + \sigma_{ij} e^e_{ij})\, dV + \int\limits_V \sigma_{ij}\varepsilon^p_{ij}\, dV - \int\limits_\Sigma \frac{\sigma^+_{ij} + \sigma^-_{ij}}{2}\,[e^p_{ij}]\, dS = \int\limits_S \sigma_{ij} v_i N_j\, dS + \int\limits_V \rho\chi_i v_i\, dV. \tag{2.6}$$

Equality (2.6) is an extension of the velocity equation from [12, 13] to the case of propagation of a strain discontinuity surface in a body. Equality (2.6) is easy to extend to the case where several, rather than one, discontinuity surfaces are present in volume $V$. We note that equality (2.6) is a consequence of the conservation laws under the assumption that the strains admitted by the medium are small and the relationship between the stresses and reversible strains is linear; the plastic flow model is insignificant in this case.

According to equality (2.6), the power of the external action on the body (the right side of the equality) is expended in changing the kinetic and potential energy of the body (the first terms on the left side) and in producing internal energy in the volume and on the discontinuity surfaces (the second and the third terms on the left side, respectively). According to the mechanical meaning, the last two terms on the right side of Eq. (2.6) should be positive. From equalities (2.3) and (2.5), it follows that the mechanical meaning of the last term on the left side (2.6) is an irreversible source of internal energy on the surface $\Sigma(t)$. The equation of possible powers which is similar to (2.6) but does not contain the term due to the internal-energy source on the irreversible-strain discontinuity surface is commonly used to prove that the stress and stress-rate distributions are unique in elastoplastic bodies [13]. To extend the conclusions of [13] to the case of presence of irreversible-strain surfaces in elastoplastic body, one needs additional conditions. As such a condition we use the extension of the Mises maximum principle to the dissipative process on $\Sigma(t)$: as the irreversible strains on the discontinuity surface vary from the value $e^{p+}_{ij}$ to the value $e^{p-}_{ij}$, the stresses vary so that the product $(\sigma^+_{ij} + \sigma^-_{ij})[e^p_{ij}]$ takes the maximum value for all possible stresses that satisfy the conditions $f^{(s)}(\sigma^+_{ij}, \tau^+) \leqslant k$ and $f^{(s)}(\sigma^-_{ij}, \tau^-) = k$. This condition, which is a natural extension of the Mises maximum principle to shock-wave processes ($[e^p_{ij}] \neq 0$) is the classical maximum principle when $\sigma^+_{ij}$ tends to the value $\sigma_{ij}^-$ and $[e_{ij}^p]$ becomes $\dot{e}_{ij}^p = \varepsilon_{ij}^p$. It has been shown [4] that the extremality condition of the stress–strain states on the discontinuity surface, which is similar to the extension of the maximum principle given above, implies invariance of the principal axes of the stress tensor across the discontinuity surface $\Sigma(t)$. This result underlies all conclusions [4] on the existence conditions and propagation mechanisms of strong-discontinuity surfaces. Later, the invariance of the principal directions across a strong discontinuity surface was inferred using the extreme conditions of shock-wave transition formulated differently [5–8, 14].

**3. Discontinuities For the Ishlinskii–Ivlev Yield Conditions.** To write the relations on the discontinuity surface, we assume that it is a thin transition layer of thickness $2h$ in which the medium has only plastic properties. We assume that the stress state corresponds to the face of the Ishlinskii–Ivlev pyramid (1.5) over the entire transition layer. In this case, for all cases of the stress state, the following relations hold:

$$(2+q)\sigma_1 + (q-1)(\sigma_2+\sigma_3) = 2k; \tag{3.1}$$

$$(q-2)\sigma_1 + (q+1)(\sigma_2+\sigma_3) = 2k. \tag{3.2}$$

The choice of one of the equalities (3.1) or (3.2) concretizes the face. The solutions on the other faces are the same up to the redefinition of the principal stresses.

Integrating the equations of motion (1.6) in the transition layer with the use of the associated plastic flow law and other dependences of the model, it is possible to obtain relations for discontinuities; for example, if the condition (3.1) is satisfied, we have

$$\begin{gathered} -c([\sigma_1]l_il_j + [\sigma_2]m_im_j + [\sigma_3]n_in_j) = (\lambda[v_i] - \eta\Phi)\delta_{ij} + \mu([v_i]\nu_j + [v_j]\nu_i) - 3\mu l_il_j\Phi, \\ (\mu - \rho c^2)[v_i] - (\eta\nu_i + 3\mu l_il_3)\Phi + (\lambda+\mu)[v_3]\nu_i = 0, \\ (2+q)[\sigma_1] + (q-1)([\sigma_2]+[\sigma_3]) = 0, \end{gathered} \tag{3.3}$$

where

$$\Phi = \int\limits_{-h}^{h} \psi\, dx_3, \qquad \eta = \frac{3}{2}\lambda q + \mu(q-1), \qquad l_il_j + m_im_j + n_in_j = \delta_{ij},$$

$$\sigma_{ij} = \sigma_1 l_il_j + \sigma_2 m_im_j + \sigma_3 n_in_j, \qquad \varepsilon_{ij}^p = \varepsilon_1^p l_il_j + \varepsilon_2^p m_im_j + \varepsilon_3^p n_in_j.$$

In relations (3.3) and everywhere below, the coordinate system is chosen so that the $x_3$ axis is normal to the surface $\Sigma(t)$ and the $x_1$ and $x_2$ axes satisfy the condition $l_2 = 0$.

With the use of relation (3.2), Eq. (3.3) and the system of discontinuity equations imply the possibility of the existence of longitudinal dissipative discontinuities ($[v_1] = [v_2] = 0$ and $[v_3] \neq 0$). Such discontinuities exist if the normal to the discontinuity surface is collinear to one of the principal axes of the stress tensor. For the velocities of motion of these surfaces, the following relations are valid:

$$c^2 = \frac{\mu}{3\rho}\,\frac{q(3\lambda+6\mu+4\eta)+6\lambda+12\mu+2\eta}{q(\eta+\mu)+2\mu}, \qquad c^2 = \frac{2\mu}{3\rho}\,\frac{(1-q)(3\lambda-2\eta)}{q(\eta+\mu)+2\mu},$$

$$c^2 = \frac{\mu}{3\rho}\,\frac{q(-3\lambda+2\mu+4\eta)+6\lambda+8\mu-2\eta}{q(\eta+\mu)+2\mu}, \qquad c^2 = \frac{2\mu}{3\rho}\,\frac{(1+q)(3\lambda+4\mu+2\eta)}{q(\eta+\mu)+2\mu}.$$

Eliminating the plastic compressibility ($q = 0$) from these relation, we obtain the well-known values $c^2 = (3\lambda + 5\mu)/(3\rho)$ and $c^2 = (3\lambda+2\mu)/(3\rho)$.

We consider the case of the existence of combined dissipative discontinuities ($[v_1] \neq 0$, $[v_2] = 0$ and $[v_3] \neq 0$) which move at velocities lower than the velocities of elastic shear waves. From system (3.3) and its analog for the case of using relation (3.2), we have

$$[v_2] = 0, \qquad [v_3] = \frac{1}{2}\,[v_1]\,\frac{1-2l_1^2}{l_1l_3}. \tag{3.4}$$

For the combined discontinuities to exist, it is necessary that the conditions $m_2 = \pm 1$ or $n_2 = \pm 1$ be satisfied, i.e., it is necessary that the normal to the discontinuity surfaces should be orthogonal to one of the indicated principal axes of the prestress tensor. The propagation velocities of such discontinuities can have the following values:

$$c^2 = \frac{\mu}{\rho}\,\frac{(1-q)(3\lambda-2\eta)}{q(-3\lambda+2\eta)+12\lambda+9\mu-2\eta} \quad \text{at} \quad \Phi = 2[v_3]\,\frac{\lambda+\mu}{3\mu+2\eta},$$

$$c^2 = \frac{\mu}{\rho}\,\frac{(1+q)(3\lambda+4\mu+2\eta)}{q(3\lambda+4\mu+2\eta)+12\lambda+13\mu+2\eta} \quad \text{at} \quad \Phi = 2[v_3]\,\frac{\lambda+\mu}{\mu+2\eta}.$$

We note that, depending on the type of preliminary stress state, this discontinuity can lead to an increase or a decrease in the strain due to shape and volume changes. As $q \to 0$ (transition to plastic incompressibility), the velocities of these discontinuities have the single value

$$c^2 = \frac{\mu}{\rho}\,\frac{3\lambda+2\mu}{12\lambda+11\mu}.$$

Let the relations for the plastic flow in the transition layer be closed by the condition specifying that the stress state belongs to the face of the Ishlinskii–Ivlev pyramid. With accuracy up to redefinition of the principal stresses, this condition can be written as

$$\sigma_1 - \sigma + q\sigma = \sigma - \sigma_3 + q\sigma = 2k/3.$$

In this case, the system of discontinuity equations becomes

$$-c([\sigma_1](l_il_j + m_im_j/2) + [\sigma_3](m_im_j/2 + n_in_j)) = \lambda[v_i]\delta_{ij} + \mu([v_i]\nu_j + [v_j]\nu_i) + \beta_{ij},$$

$$(\mu - \rho c^2)[v_i] + (\lambda+\mu)[v_3]\nu_i + \beta_{i3} = 0, \tag{3.5}$$

$$(q+1)[\sigma_1] + (q-1)[\sigma_3] = 0,$$

where

$$\beta_{ij} = -(3/2)\lambda q\delta_{ij}(\Phi_1+\Phi_2) - \mu\delta_{ij}((q-1)\Phi_1 + (q+1)\Phi_2) - 3\mu(\Phi_1 l_il_j - \Phi_2 n_in_j),$$

$$\Phi_1 = \int\limits_{-h}^{h} \psi_1\,dx_3, \qquad \Phi_2 = \int\limits_{-h}^{h} \psi_2\,dx_3.$$

From (3.5), it follows that, under these conditions, only the longitudinal discontinuities of the surfaces ($[v_1] = [v_2] = 0$ and $[v_3] \neq 0$) can exist. For such dissipative discontinuities to exist, it is necessary that the normal to the discontinuity surfaces be collinear to one of the principal axes of the stress tensor. In this case, the result is similar to the result obtained with irreversible compressibility neglected [14]. The difference is only due to the refinement of the value for the propagation velocities of such discontinuities:

$$c^2 = \frac{\mu}{\rho}\,\frac{3\lambda+2\mu}{q^2(3\lambda+2\mu)+3\mu}, \quad c^2 = \frac{\mu}{\rho}\,\frac{(3\lambda+2\mu)(q-1)^2}{q^2(3\lambda+2\mu)+3\mu}, \quad c^2 = \frac{\mu}{\rho}\,\frac{(3\lambda+2\mu)(q+1)^2}{q^2(3\lambda+2\mu)+3\mu}.$$

Eliminating plastic compressibility ($q = 0$) from these relation, we obtain the well-known value $c^2 = (3\lambda+2\mu)/(3\rho)$ [4, 7].

If the stress state ahead of the discontinuity surface and in the transition layer correspond to the edge of the pyramid formed by the intersection of its face and base, conditions (3.1) and (3.2) should be supplemented by the requirement $\sigma = -\varphi(\tau)$. Then, the system of discontinuity equations [an analog of (3.3)] includes the experimentally obtained function $\varphi(\tau)$, which is considered known in this paper. However, it should be noted that the conditions on the preliminary stress state for the occurrence of dissipative discontinuities remain the same as those on the edge of the pyramid. In the case of a combined discontinuity, relations (3.4) remain valid. The transition of the stress state from the face to the edge influences the propagation velocities of possible discontinuities. Thus, in the case of a longitudinal discontinuity which also exists if the normal to the discontinuity surface is collinear to one of the principal axes of the stress tensor, we have

$$c^2 = \frac{4\mu}{\rho} \frac{A\varphi'(3\lambda+2\mu)(q-1)^2}{\theta}, \qquad c^2 = \frac{4\mu}{\rho} \frac{(3\lambda+2\mu)(A\varphi' q(q+1)-\mu)+A\varphi'(3\lambda+5\mu)}{\theta},$$

$$c^2 = \frac{4\mu}{\rho} \frac{A\varphi'(3\lambda+2\mu)(q+1)^2}{\theta}, \qquad c^2 = \frac{4\mu}{\rho} \frac{(3\lambda+2\mu)(A\varphi' q(q-1)-\mu)-A\varphi'(3\lambda+5\mu)}{\theta},$$

$$\theta = 3q^2 A\varphi'(3\lambda+2\mu) - 8\mu^2 - 12\mu(\lambda - A\varphi'),$$

where the parameter $A$ is an invariant tensor $A_{ij}$ (1.3) which depends on the strain history [15].

In the case of a combined discontinuity which arises if one of the relations $m_2 = \pm 1$ or $n_2 = \pm 1$ is satisfied, we obtain two values for the propagation velocities of the possible discontinuities, depending on whether face (3.1) or (3.2) is used, respectively:

$$c^2 = \frac{\mu}{\rho} \frac{A\varphi'(3\lambda+2\mu)(q-1)^2}{(3\lambda+2\mu)(q^2A\varphi' - 2qA\varphi' - \mu) + A\varphi'(12\lambda+11\mu)},$$

$$c^2 = \frac{\mu}{\rho} \frac{A\varphi'(3\lambda+2\mu)(q+1)^2}{(3\lambda+2\mu)(q^2A\varphi' + 2qA\varphi' - \mu) + A\varphi'(12\lambda+11\mu)}.$$

Let the stress state of the medium correspond to the singular point of the pyramid formed by intersection of its edge and bottom. In the case considered, only longitudinal discontinuities can propagate; they exist only if the normal to the discontinuity surface is collinear to one of the principal axes of the stress tensor. The propagation velocities of such discontinuities are given by the relations

$$c^2 = \frac{\mu}{\rho} \frac{A\varphi'(3\lambda+2\mu)}{(3\lambda+2\mu)(q^2A\varphi' - \mu) + 3\mu A\varphi'}, \qquad c^2 = \frac{\mu}{\rho} \frac{A\varphi'(3\lambda+2\mu)(q-1)^2}{(3\lambda+2\mu)(q^2A\varphi' - \mu) + 3\mu A\varphi'}.$$

**4. Discontinuities For the Coulomb–Mohr Yield Conditions.** Let the stress state correspond to the face of the Coulomb–Mohr pyramid (1.4) over the entire transition layer. The equation of each face (up to redefinition of the principal stresses) can be written as

$$(\sigma_1 - \sigma_3)/2 + q\sigma = k. \tag{4.1}$$

In this case, the system of discontinuity equations on $\Sigma(t)$ becomes

$$-c([\sigma_1]l_il_j + [\sigma_2]m_im_j + [\sigma_3]n_in_j) = (\lambda[v_3] - (\lambda + 2\mu/3)q\Phi)\delta_{ij} + \mu([v_i]\nu_j + [v_j]\nu_i) + \mu\Phi(n_in_j - l_il_j),$$

$$(\mu - \rho c^2)[v_i] + (\lambda+\mu)[v_3]\nu_i - q(\lambda + 2\mu/3)\Phi\nu_i + \mu\Phi(n_in_3 - l_il_3) = 0, \tag{4.2}$$

$$(1/2 + q/3)[\sigma_1] + q[\sigma_2]/3 + (-1/2 + q/3)[\sigma_3] = 0.$$

As in the case of Ishlinskii–Ivlev pyramid, system (4.2) admits the existence of longitudinal and combined discontinuities.

The propagation velocities of longitudinal discontinuities ($[v_1] = [v_2] = 0$ and $[v_3] \neq 0$) take the values

$$c^2 = \frac{\mu}{3\rho} \frac{2q(2q+3)(3\lambda+2\mu) + 9(\lambda+\mu)}{q^2(3\lambda+2\mu) + 3\mu}, \qquad c^2 = \frac{\mu}{3\rho} \frac{4q^2(3\lambda+2\mu) + 9(\lambda+\mu)}{q^2(3\lambda+2\mu) + 3\mu},$$

$$c^2 = \frac{\mu}{3\rho} \frac{2q(2q-3)(3\lambda+2\mu) + 9(\lambda+\mu)}{q^2(3\lambda+2\mu) + 3\mu}.$$

Ignoring irreversible compressibility ($q = 0$), we obtain the well-known value $c^2 = (\lambda+\mu)/\rho$ [14].

The constraints on the jumps of the displacement velocities $[v_i]$ on the surfaces of the combined discontinuities become

$$[v_2] = 0, \qquad [v_3] = \frac{1}{2}[v_1]\frac{1-2n_2^2}{n_2n_3} \quad \text{or} \quad [v_3] = \frac{1}{2}[v_1]\frac{1-2l_1^2}{l_1l_3}. \tag{4.3}$$

Thus, two of the three possible propagation velocities of combined discontinuities are given by

$$c^2 = \frac{\mu}{\rho} \frac{(4q^2+12q+9)(3\lambda+2\mu)}{4q(q+3)(3\lambda+2\mu) + 9(4\lambda+3\mu)}, \qquad c^2 = \frac{\mu}{\rho} \frac{(4q^2-12q+9)(3\lambda+2\mu)}{4q(q-3)(3\lambda+2\mu) + 9(4\lambda+3\mu)}.$$

Ignoring irreversible compressibility ($q = 0$), we obtain the well-known value $c^2 = (\mu/\rho)(3\lambda+2\mu)/(4\lambda+3\mu)$ [14].

In the case of a combined discontinuity, the third velocity admitted by system (4.2) is of interest since it is possible only in a plastically compressible medium ($q \neq 0$):

$$c^2 = \frac{\mu}{\rho} \frac{q^2(3\lambda + 2\mu)}{q^2(3\lambda + 2\mu) + 9(\lambda + \mu)} \quad \text{for} \quad \Phi = 3[v_3] \frac{\lambda + \mu}{q(3\lambda + 2\mu)}.$$

Let the stress state in the transition layer correspond to the edge of the Coulomb–Mohr pyramid (1.4). All possibilities that arise in this case are exhausted (up to redefinition of the principal values of the stress tensor) by the conditions

$$(\sigma_1 - \sigma_3)/2 + q\sigma = (\sigma_1 - \sigma_2)/2 + q\sigma = k. \tag{4.4}$$

In this case, the system of discontinuity equations becomes

$$\begin{gathered} -c([\sigma_1]l_i l_j + [\sigma_3](m_i m_j + n_i n_j)) = \lambda[v_3]\delta_{ij} + \mu[v_j]\nu_i + \beta_{ij}, \\ (\mu - \rho c^2)[v_i] + (\lambda + \mu)[v_3]\nu_i + \beta_{i3} = 0, \\ (2q + 3)[\sigma_1] + (4q - 3)[\sigma_3] = 0, \end{gathered} \tag{4.5}$$

where

$$\beta_{ij} = -q\delta_{ij}(\lambda + 2\mu/3)(\Phi_1 + \Phi_2) + \mu\Phi_1(n_i n_j - l_i l_j) + \mu\Phi_2(m_i m_j - l_i l_j).$$

System (4.5) admits the existence of only longitudinal dissipative discontinuities propagating at the velocities

$$c^2 = \frac{\mu}{3\rho} \frac{(3\lambda + 2\mu)(4q^2 + 12q + 9)}{4q^2(3\lambda + 2\mu) + 9\mu}, \qquad c^2 = \frac{\mu}{3\rho} \frac{(3\lambda + 2\mu)(16q^2 - 24q + 9)}{q^2(3\lambda + 2\mu) + 3\mu}.$$

For $q = 0$, these relation lead to the well-known values $c^2 = (\lambda + 2\mu/3)/\rho$ [4, 14]. As above, the necessary condition for the existence of such discontinuities is the collinearity of the normal $\boldsymbol{\nu}$ to one of the principal axes of the stress tensor.

Let the stress state of the medium correspond to the edge of the Coulomb–Mohr pyramid formed by the intersection of its face and base. To obtain the system of discontinuity equations for this case, it is sufficient to supplement relation (4.1) by the equality $\sigma = -\varphi(\tau)$. In this case, the conditions on the prestress state for the occurrence of dissipative discontinuities remain the same as on the face of the pyramid. For the velocity jumps $[v_i]$ on the combined discontinuity surface, relations (4.3) remain valid; the propagation velocities of such discontinuities take the values

$$c^2 = -\frac{\mu}{\rho} \frac{A\varphi'(3\lambda + 2\mu)(4q^2 + 12q + 9)}{(3\lambda + 2\mu)(-4q^2 A\varphi' - 12qA\varphi' + \mu) + 9A\varphi'(4\lambda + 3\mu)},$$

$$c^2 = \frac{\mu}{\rho} \frac{A\varphi'(3\lambda + 2\mu)(4q^2 - 12q + 9)}{(3\lambda + 2\mu)(4q^2 A\varphi' - 12qA\varphi' - \mu) + 9A\varphi'(4\lambda + 3\mu)},$$

$$c^2 = -\frac{\mu}{\rho} \frac{A\varphi' q^2(3\lambda + 2\mu)}{(3\lambda + 2\mu)(-q^2 A\varphi' + \mu) - 9A\varphi'(\lambda + \mu)}.$$

For the longitudinal-discontinuity surfaces, we have

$$c^2 = \frac{\mu}{3\rho} \frac{(3\lambda + 2\mu)(4q^2 A\varphi' + 6qA\varphi' - \mu) + 9A\varphi'(\lambda + \mu)}{(q^2 A\varphi' - \mu)(3\lambda + 2\mu) + 3\mu A\varphi'},$$

$$c^2 = \frac{\mu}{3\rho} \frac{(3\lambda + 2\mu)(4q^2 A\varphi' - 6qA\varphi' - \mu) + 9A\varphi'(\lambda + \mu)}{(q^2 A\varphi' - \mu)(3\lambda + 2\mu) + 3\mu A\varphi'},$$

$$c^2 = \frac{\mu}{3\rho} \frac{(3\lambda + 2\mu)(4q^2 A\varphi' - 4\mu) + 9A\varphi'(\lambda + 2\mu)}{(q^2 A\varphi' - \mu)(3\lambda + 2\mu) + 3\mu A\varphi'}.$$

If the stress state of the medium corresponds to the singular point of the pyramid (1.4) formed by the intersection of its edge and bottom, the system of discontinuity equations is obtained by setting $\sigma = -\varphi(\tau)$ in relations (4.4). In this case, only longitudinal discontinuities can propagate at the velocities

$$c^2 = \frac{\mu}{3\rho}\,\frac{A\varphi'(3\lambda+2\mu)(4q^2-12q+9)}{(3\lambda+2\mu)(4q^2A\varphi'-\mu)+9\mu A\varphi'},$$

$$c^2 = \frac{\mu}{3\rho}\,\frac{A\varphi'(3\lambda+2\mu)(16q^2+24q+9)}{(3\lambda+2\mu)(4q^2A\varphi'-\mu)+9\mu A\varphi'}.$$

**Conclusions.** Thus, accounting for plastic compressibility not only allows one to refine the propagation velocities of discontinuity surfaces but also leads to a considerable increase in the number of possible discontinuities that can propagate at different velocities. In other words, if one discontinuity surface arises in a plastically incompressible medium, accounting for compressibility leads to the occurrence of a combination of two discontinuities moving at different velocities. However, the conditions for the occurrence of discontinuities used as constraints on prestress mainly coincide with the similar restrictions formulated in the classical Prandtl–Reuss models [4, 14]. In this case, they are proved to be identical for both the cases of the Coulomb–Mohr pyramid and the Ishlinskii–Ivlev pyramid.

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